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# One-Step A-Stable Implicit Block Method for Approximating the Solutions of First Order Stiff Ordinary Differential Equations

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**Abstract—** A self-starting A-stable implicit linear multistep block method for approximating stiff initial value problems (IVPs) in ordinary differential equations (ODEs) is developed. The construction was carried out by pairing a three-step top order method (TOM) and a four-step linear multistep method and shifting each equation four times to form a block method that generates approximations on ten grid points simultaneously. The implementation of the method on some stiff ODEs confirms its efficiency.

**Keywords—** A-Stable, Block Method, Linear Multistep Method (LMM) and Top Order Methods (TOM).

## 1.0 INTRODUCTION

First order ordinary differential equations (ODEs) of the form

$$\left.\begin{aligned} y'(t) &= f(x, y(t)) \\ y(t_0) &= y_0 \end{aligned}\right\} \qquad (1.1)$$

represent important mathematical models of real-world phenomena. They are useful tools in the formation of epidemiological models [ 1, 2, 3], dynamic systems [4], chemical reactions, electrical circuits [5] etc. To this end, first order ordinary differential equations are applied in different fields such as physical sciences, health sciences, social sciences and engineering.

Over the years, many researchers have concentrated on obtaining solutions to equation (1.1) because an $nth$ order ordinary differential equation can be solved by reducing it to a system of first order ODEs which are easier to programme.

some first order ODEs or systems are stiff. “Stiffness occurs in differential equations where two or more different time scales of the independent variables on which the dependent variables are changing” [6]. A stiff ODE can either be linear or nonlinear.

In linear problems, stiffness is caused by eigenvalues of large negative values. The degree of stiffness is measured using the ratio $SR = \frac{max|\lambda|}{min|\lambda|}$, where $\lambda$ denote an eigenvalue [6].

 

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Although, there are many analytical and semi analytical methods of solving stiff ODEs, especially linear stiff ODEs, numerical methods remain valuable methods for dealing with nonlinear problems and for investigating and analyzing simple special cases [7]. In recent years, with the invention of software such as MAPLE, MATLAB and MATHEMATICA, many researchers have developed A-stable numerical methods for handling stiff ODEs.

Block methods provide very high accurate methods that are absolute stable (A-stable) and circumvent Dalqhist barrier [8], they were introduced to improve stability of methods [9].

Since most block methods are constructed using linear multistep methods, they also provide $k-1$ starting values and simultaneously generate $k$ approximations [8]. For more on the advantages of block methods, the following can be consulted [10, 11, 5, 12, 13].

The aim of this paper is to develop a 6th order implicit linear multistep method (LMM) and combine it with the top order method (TOM) of order 6 to form a pair which is used to construct a one-step implicit block method for approximating stiff ODEs.

The rest of the paper is organized as follows, Section 2 deals with the formulation and analysis of the implicit block method, the implementation is carried out in Section 3, while the discussion and conclusion are provided in Section 4 and 5

## 2.0 METHODS

In this Section, we derive a symmetric linear multistep scheme of order $(k+2)=6$, which is combined with a top order method (TOM) of order $2k=6$, to form a block method that is self-starting. The pairs are shifted forward simultaneously four times to give a set of ten equations which are solved to obtain the values of the unknown, $y_n, y_{n+1}, y_{n+2}, \cdots, y_{n+10}$.

### *2.1 Derivation of a Symmetric Method of Order Six (6)*

We derive a symmetric scheme of order $(k+2)=6$, where $k=4$. To this end, consider the polynomial function represented by $f(x_n, y(x_n))$.

Let

$$f\big(x_n, y(x_n)\big) = P_n(x_n) = \sum_{k=0}^{n} \binom{r}{k} \Delta^k y_n \tag{2.1}$$

where

$x = x_n + rh, dx = hdr$

and

$$r = \frac{x - x_n}{h}, \tag{2.2}$$

$\Delta$ denotes the forward operator defined by $\Delta^k y_n = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_n$.

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Let the interpolating polynomial $P_n(x_n)$ be defined by,

$$\left.\begin{aligned} P_n(x_n) = f_n + r\Delta f_n + \frac{r(r-1)}{2!}\Delta^2 f_n + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_n + \\ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_n + \cdots + f_n\frac{r(r-1)\dots(r-n+1)}{n!}\Delta^n f_n \end{aligned}\right\} \quad (2.3)$$

Consider the general well-posed first order initial value problem (IVP),

$$\frac{dy(x_n)}{dx} = f(x_n, y(x_n)), \quad y(x_0) = y_0. \quad (2.4)$$

The derivation of the symmetric scheme of order $(k+2)$, for $k = 4$ is done in two stages, after which the final scheme is obtained by summing the results obtained from the two stages.

***STAGE 1***

Integrating equation (2.4) from $x_{n+3}$ to $x_{n+4}$ we have,

$$\int_{x_{n+3}}^{x_{n+4}} dy(x_n) = \int_{x_{n+3}}^{x_{n+4}} f((x_n, y(x_n))dx \quad (2.5)$$

Using the first five terms of equation (2.3) and substituting in equation (2.5) and further simplification gives,

$$\left.\begin{aligned} &y(x_{n+4}) - y(x_{n+3}) = \\ &\int_{x_{n+3}}^{x_{n+4}} \left[f_n + r\Delta f_n + \frac{r(r-1)}{2!}\Delta^2 f_n + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_n + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_n\right] h\,dr \end{aligned}\right\} (2.6)$$

or

$$\left.\begin{aligned} &y(x_{n+4}) - y(x_{n+3}) = \\ &h\left[rf_n + \frac{r^2}{2}\Delta f_n + \frac{2r^3 - 3r^2}{12}\Delta^2 f_n + \frac{r^4 - 4r^3 + 4r^2}{24}\Delta^3 f_n + \frac{6r^5 - 45r^4 + 110r^3 - 90r^2}{720}\Delta^4 f_n\right]_3^4 \end{aligned}\right\} (2.7)$$

Substituting the limits and simplifying the R.H.S. of equation (2.7) gives,

$$h\left\{\left[4f_n + 8\Delta f_n + \frac{20}{3}\Delta^2 f_n + \frac{8}{3}\Delta^3 f_n + \frac{14}{45}\Delta^4 f_n\right] - \left[3f_n + \frac{9}{2}\Delta f_n + \frac{9}{4}\Delta^2 f_n + \frac{3}{8}\Delta^3 f_n - \frac{3}{80}\Delta^4 f_n\right]\right\}$$

$$\boldsymbol{\therefore y_{n+4} - y_{n+3} = h\left[-\frac{19}{720}f_n + \frac{53}{360}f_{n+1} - \frac{11}{30}f_{n+2} + \frac{323}{360}f_{n+3} + \frac{251}{720}f_{n+4}\right]} \quad \mathbf{(2.8)}$$

***STAGE 2***

Integrating equation (2.4) from $x_n$ to $x_{n+1}$ gives,

$$\int_{x_n}^{x_{n+1}} dy(x_n) = \int_{x_n}^{x_{n+1}} f(x_n, y(x_n))dx. \quad (2.9)$$

 

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Using the first five terms of equation (2.3) and substituting in equation (2.9) and further simplification yields,

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} \left[ f_n + r\Delta f_n + \frac{r(r-1)}{2!}\Delta^2 f_n + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_n \right] hdr$$

$$\left. \begin{aligned} & y(x_{n+1}) - y(x_n) = \\ & h\left[ rf_n + \frac{r^2}{2}\Delta f_n + \frac{2r^3 - 3r^2}{12}\Delta^2 f_n + \frac{r^4 - 4r^3 + 4r^2}{24}\Delta^3 f_n + \frac{6r^5 - 45r^4 + 110r^3 - 90r^2}{720}\Delta^4 f_n \right]_0^1 \end{aligned} \right\} (2.10)$$

Substituting the limits and simplifying the R.H.S. of equation (2.12) gives,

$$h\left[ f_n + \frac{1}{2}\Delta f_n - \frac{1}{12}\Delta^2 f_n + \frac{1}{24}\Delta^3 f_n - \frac{19}{720}\Delta^4 f_n \right]. \tag{2.11}$$

Simplifying equation (2.11) gives,

$$\therefore \boldsymbol{y_{n+1} - y_n = \left\{ \frac{251}{720} f_n + \frac{323}{360} f_{n+1} - \frac{11}{30} f_{n+2} + \frac{53}{360} f_{n+3} - \frac{19}{720} f_{n+4} \right\}}. \qquad (\mathbf{2.12})$$

Adding equation (2.8) and equation (2.12) results to:

$$\boldsymbol{y_{n+4} - y_{n+3} + y_{n+1} - y_n = \frac{h}{90}[29f_n + 94f_{n+1} - 66f_{n+2} + 94f_{n+3} + 29f_{n+4}]} \qquad (\mathbf{2.13})$$

Equation (2.13) is a symmetric scheme of order $k = 4$.

***2.2 Six Order Top Order Method (TOM)***

Consider the numerical scheme,

$$\boldsymbol{-\frac{11}{60} y_n - \frac{9}{20} y_{n+1} + \frac{9}{10} y_{n+2} + \frac{11}{60} y_{n+3} = h\left[ \frac{1}{20} f_n + \frac{9}{20} f_{n+1} + \frac{9}{20} f_{n+2} + \frac{1}{20} f_{n+3} \right]} (\mathbf{2.14})$$

Equation (2.14) is a 3-step TOM formula of order 6 which was considered by [12]. Analysis of the TOMs show that they are higher order accurate methods of order $p = 2k$ but are unstable methods.

Equation (2.14) is combined with the derived symmetric method, equation (2.13) to develop a block method for the solution of first order stiff ordinary differential equations.

## *2.3 Construction of the Block Method*

The block method is developed by replacing $n$ by $n + 3$ in equation (2.14) and combining with equation (2.13) to form a pair which is shifted four times as presented in block formation.

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$$A_0=\begin{pmatrix}0&0&0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},\ B_0=\begin{pmatrix}0&0&0&0&0&0&0&0&0&\frac{29}{90}\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix}$$

$$A_1=\begin{pmatrix}1&0&-1&1&0&0&0&0&0&0\\0&0&\frac{-11}{60}&\frac{-9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0&0\\-1&1&0&-1&1&0&0&0&0&0\\0&0&0&\frac{-11}{60}&\frac{-9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0\\0&-1&1&0&-1&1&0&0&0&0\\0&0&0&0&\frac{-11}{60}&\frac{-9}{20}&\frac{9}{20}&\frac{11}{60}&0&0\\0&0&-1&1&0&-1&1&0&0&0\\0&0&0&0&0&\frac{-11}{60}&\frac{-9}{20}&\frac{9}{20}&\frac{11}{60}&0\\0&0&0&-1&1&0&-1&1&0&0\\0&0&0&0&0&0&\frac{-11}{60}&\frac{-9}{20}&\frac{9}{20}&\frac{11}{60}\end{pmatrix}$$

$$B_1=\begin{pmatrix}\frac{94}{90}&-\frac{66}{90}&\frac{94}{90}&\frac{29}{90}&0&0&0&0&0&0\\0&0&\frac{1}{20}&\frac{9}{20}&\frac{9}{20}&\frac{1}{20}&0&0&0&0\\\frac{29}{90}&\frac{94}{90}&-\frac{66}{90}&\frac{94}{90}&\frac{29}{90}&0&0&0&0&0\\0&0&0&\frac{1}{20}&\frac{9}{20}&\frac{9}{20}&\frac{1}{20}&0&0&0\\0&\frac{29}{90}&\frac{94}{90}&-\frac{66}{90}&\frac{94}{90}&\frac{29}{90}&0&0&0&0\\0&0&0&0&\frac{1}{20}&\frac{9}{20}&\frac{9}{20}&\frac{1}{20}&0&0\\0&0&\frac{29}{90}&\frac{94}{90}&-\frac{66}{90}&\frac{94}{90}&\frac{29}{90}&0&0&0\\0&0&0&0&0&\frac{1}{20}&\frac{9}{20}&\frac{9}{20}&\frac{1}{20}&0\\0&0&0&\frac{29}{90}&\frac{94}{90}&-\frac{66}{90}&\frac{94}{90}&\frac{29}{90}&0&0\\0&0&0&0&0&0&\frac{1}{20}&\frac{9}{20}&\frac{9}{20}&\frac{1}{20}\end{pmatrix}$$

### *2.4.1 The Order and Consistency of the Block Method*

**Order of the Block Method**

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The order of the block method is carried out following the method of block formation and is confirmed to be of order

$p = (6, \quad 6, \quad 6, \quad 6, \quad 6, \quad 6, \quad 6, \quad 6, \quad 6, \quad 6)^T$

with error constant

$$p_{n+1} = \left(\frac{271}{60480} \quad -\frac{37}{3780} \quad -\frac{331}{20160} \quad -\frac{29}{945} \quad -\frac{317}{12096} \quad -\frac{37}{1260} \quad -\frac{43}{1728} \quad -\frac{37}{945} \quad \frac{2549}{20160} \quad -\frac{12079}{91476}\right)^T$$

## *2.4.2 Convergence Analysis*

Consider the general convergence analysis which is given by the matrix formation of the block method are given as

$A_1 Y_{\omega+1} + A_0 Y_\omega = hB_0 F_\omega + hB_1 F_{\omega+1}$ where $A_1, A_0, B_0$ and $B_1$ are $M \times M$ matrices

Multiplying through by $A_1^{-1}$ gives

$A_1^{-1}A_1 y_{n+1} + A_1^{-1}A_0 y_n = hA_1^{-1}B_0 f_n + hA_1^{-1}B_1 f_{n+1}$

or

$$Iy_{n+1} + Cy_n = hDf_n + hEf_{n+1} \qquad (2.15)$$

where $I = A_1^{-1}A_1, C = A_1^{-1}A_0, D = A_1^{-1}B_0$ and $E = A_1^{-1}B_1$.

Using the test function $y' = \lambda y$ and substituting $\lambda y_n$ for $f_n$ and $\lambda y_{n+1}$ for $y_{n+1}$ leads to

$$Iy_{n+1} + Cy_n - \lambda hEy_{n+1} - \lambda hDy_n = 0 \qquad (2.16)$$

where

$y_n = f(x_n, y_n)$, and

$y_{n+1} = f(x_{n+1}, y_{n+1})$.

Substitute $Z = \lambda h$ in equation (2.16)

$$Iy_{n+1} + Cy_n - ZEy_{n+1} - ZDy_n = 0 \qquad (2.17)$$

The characteristic equation associated with (2178) is given by

$(1 - ZE)r = (ZD - C)$

or

$r = (1 - ZE)^{-1}(ZD - C)$

Let $r = A$

$\det(Ir - A) = 0$

where $A = (1 - ZE)^{-1}(ZD - C)$

$$\rho(r) = det[Ir - (1 - ZE)^{-1}(ZD - C)] \qquad (2.18)$$

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Evaluating equation (2.18) gives

$$r^9\left(r - \frac{AZ^{10} + BZ^9 + CZ^8 + DZ^7 + EZ^6 + FZ^5 + GZ^4 + HZ^3 + IZ^2 + JZ + U}{KZ^{10} - LZ^9 + MZ^8 - NZ^7 + OZ^6 - PZ^5 + QZ^4 - RZ^3 + SZ^2 - TZ + U}\right) = 0 \quad (2.19)$$

where

| | | |
|---|---|---|
| $A = 749376414$ | $B = 9072748368$ | $C = 45891413805$ |
| $D = 134757262436$ | $E = 281213734995$ | $F = 462030153990$ |
| $G = 582582615840$ | $H = 549874580040$ | $I = 367959931200$ |
| $J = 516274272000$ | $K = 820643724$ | $L = 9909576702$ |
| $M = 50621609595$ | $N = 154860388014$ | $0 = 336857647095$ |
| $P = 545142202410$ | $Q = 667471375440$ | $R = 608551131960$ |
| $S = 392979211200$ | $T = 161278128000$ | $U = 31755240000$ |

Recall that $Z = \lambda h$, and as $h \to 0$, equation (2.19) reduces to

$$r^9\left(r - \frac{U}{U}\right) = 0$$

or

$$r^9(r-1) = 0$$

$$\therefore r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = r_9 = 0 \text{and } r_{10} = 1$$

Since no root has modulus greater than one and $|z| = 1$ is simple, the developed block method is zero stable and the block is convergence.

### 2.4.3 Region of Absolute Stability

Equation (2.30) was plotted in MATLAB and the following region confirms that the one-step implicit block method is A-stable since the region of stability is the exterior of the circle

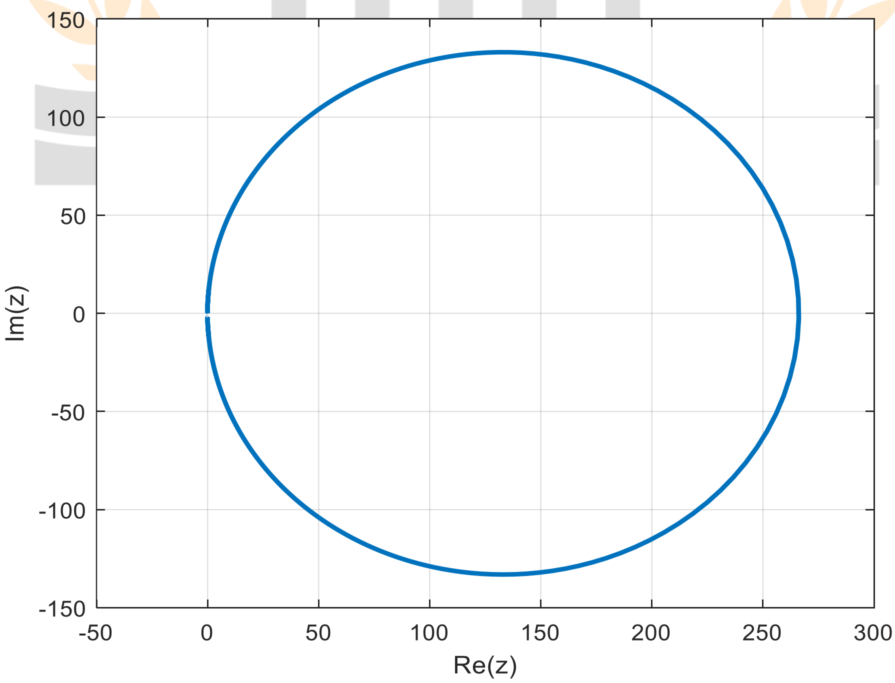


**Figure 1: Region of Absolute Stability of the Constructed Block Method**

 

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## 3.0 RESULTS

In this Section, the block method developed is tested on some first order stiff ordinary differential equations. The numerical results, analytical results and absolute errors are displayed in Tables.

**Example 1**

Consider the stiff ordinary differential equations

$y' = -10y,\ y(0) = 1,$

**Table 1: Numerical/Analytic Results for Example 1**

| N | $x_n$ | $y_n$ | $y(x_n)$ | $\lvert y_n - y(x_n)\rvert$ | Error in [8] |
|---|---|---|---|---|---|
| 0 | 0 | 1.0000000000 | 1.0000000000 | – | – |
| 1 | 0.01 | 0.9048374180 | 0.9048374180 | – | – |
| 2 | 0.02 | 0.8187307522 | 0.8187307531 | $9.0000 \times 10^{-10}$ | $7.93 \times 10^{-9}$ |
| 3 | 0.03 | 0.7408182195 | 0.7408182207 | $1.2000 \times 10^{-9}$ | |
| 4 | 0.04 | 0.6703200443 | 0.6703200460 | $1.7000 \times 10^{-9}$ | $7.36 \times 10^{-9}$ |
| 5 | 0.05 | 0.6065306583 | 0.6065306597 | $1.4000 \times 10^{-9}$ | |
| 6 | 0.06 | 0.5488116347 | 0.5488116361 | $1.4000 \times 10^{-9}$ | $6.83 \times 10^{-9}$ |
| 7 | 0.07 | 0.4965853027 | 0.4965853038 | $1.1000 \times 10^{-9}$ | |
| 8 | 0.08 | 0.4493289624 | 0.4493289641 | $1.7000 \times 10^{-9}$ | $7.03 \times 10^{-9}$ |
| 9 | 0.09 | 0.4065696604 | 0.4065696597 | $7.0000 \times 10^{-10}$ | |
| 10 | 0.10 | 0.3678794347 | 0.3678794412 | $6.5000 \times 10^{-9}$ | $2.12 \times 10^{-8}$ |

**Example 2**

Consider the system of two stiff equations

$y' = -20y - 19z, \qquad y(0) = 2,$ and

$z' = -19y - 20z, \qquad z(0) = 0,$

**Table 2: Numerical/Analytic Results for Example 2**

| n | $x_n$ | $y_n$ | $y(x_n)$ | $\lvert y_n - y(x_n)\rvert$ | $z_n$ | $z(x_n)$ | $\lvert z_n - z(x_n)\rvert$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2.0000000000 | 2.0000000000 | – | 0 | 0 | – |
| 1 | 0.01 | 1.667103289 | 1.667106708 | $3.4190 \times 10^{-6}$ | −0.3129963788 | −0.3129929592 | $3.4196 \times 10^{-6}$ |
| 2 | 0.02 | 1.438598501 | 1.438604685 | $6.1840 \times 10^{-6}$ | −0.5217988451 | −0.5217926620 | $6.1831 \times 10^{-6}$ |
| 3 | 0.03 | 1.280806104 | 1.280812475 | $6.3710 \times 10^{-6}$ | −0.6600849632 | −0.6600785922 | $6.3710 \times 10^{-6}$ |
| 4 | 0.04 | 1.170919276 | 1.170925510 | $6.2340 \times 10^{-6}$ | −0.7506596019 | −0.7506533680 | $6.2339 \times 10^{-6}$ |

 

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 0.05 | 1.093499889 | 1.093503496 | $3.6070 \times 10^{-6}$ | $-0.8089589602$ | $-0.8089553529$ | $3.6073 \times 10^{-6}$ |
| 6 | 0.06 | 1.038089291 | 1.038092172 | $2.8810 \times 10^{-6}$ | $-0.8454397756$ | $-0.8454368954$ | $2.8802 \times 10^{-6}$ |
| 7 | 0.07 | 0.9976117888 | 0.9976131096 | $1.3208 \times 10^{-6}$ | $-0.8671758508$ | $-0.8671745302$ | $1.3206 \times 10^{-6}$ |
| 8 | 0.08 | 0.9672705820 | 0.9672735148 | $2.9328 \times 10^{-6}$ | $-0.8789621102$ | $-0.8789591780$ | $2.9322 \times 10^{-6}$ |
| 9 | 0.09 | 0.9438328986 | 0.9438280997 | $4.7989 \times 10^{-6}$ | $-0.8840294720$ | $-0.8840342709$ | $4.7989 \times 10^{-6}$ |
| 10 | 0.10 | 0.9250594058 | 0.9250793294 | $1.9924 \times 10^{-5}$ | $-0.8846154292$ | $-0.8845955066$ | $1.9923 \times 10^{-5}$ |

**Example 3**

Consider the system of three stiff equations, where $t$ denotes an independent variable while $x, y$ and $z$ represent dependent variables

$$x' = -20x - 0.25y - 19.75z, \qquad x(0) = 1,$$
$$y' = \quad 20x - 20.25y + 0.25z, \qquad y(0) = 0, \qquad \text{and}$$
$$z' = \quad 20x - 19.75y - 0.25z, \qquad z(0) = -1,$$

**Table 3: Numerical/Analytic Results for Example 3**

| n | $t_n$ | $x_n$ | $x(t_n)$ | $y_n$ | $y(t_n)$ | $z_n$ | $z(t_n)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1.0000000000 | 1.0000000000 | 0 | 0 | $-1.0000000000$ | $-1.0000000000$ |
| 1 | 0.01 | 0.9800395009 | 0.9800399088 | 0.1776293511 | 0.1776292614 | $-0.8173831281$ | $-0.8173832178$ |
| 2 | 0.02 | 0.9342444613 | 0.9342452012 | 0.3168391390 | 0.3168395534 | $-0.6732106947$ | $-0.6732102802$ |
| 3 | 0.03 | 0.8739732570 | 0.8739740441 | 0.4210195506 | 0.4210202550 | $-0.5640923891$ | $-0.5640916846$ |
| 4 | 0.04 | 0.8077883452 | 0.8077890232 | 0.4947374236 | 0.4947385192 | $-0.4854612499$ | $-0.4854601540$ |
| 5 | 0.05 | 0.7418176353 | 0.7418179491 | 0.5430509762 | 0.5430518386 | $-0.4322589359$ | $-0.4322580734$ |
| 6 | 0.06 | 0.6801550359 | 0.6801551848 | 0.5710142936 | 0.5710151266 | $-0.3994312400$ | $-0.3994304070$ |
| 7 | 0.07 | 0.6252639250 | 0.6252638546 | 0.5833499086 | 0.5833504732 | $-0.3822555078$ | $-0.3822549432$ |
| 8 | 0.08 | 0.5783522634 | 0.5783522936 | 0.5842467243 | 0.5842475754 | $-0.3765427150$ | $-0.3765418638$ |
| 9 | 0.09 | 0.5397093301 | 0.5397085794 | 0.5772654202 | 0.5772648331 | $-0.3787320615$ | $-0.3787326487$ |
| 10 | 0.10 | 0.5089832982 | 0.5089850496 | 0.5653008826 | 0.5653043996 | $-0.3859285425$ | $-0.3859250248$ |

**Table 4: Absolute Errors for Example 3**

| n | $t_n$ | $\lvert x_n - x(t_n)\rvert$ | $\lvert y_n - y(t_n)\rvert$ | $\lvert z_n - z(t_n)\rvert$ |
|---|---|---|---|---|
| 0 | 0 | – | – | – |
| 1 | 0.01 | $4.0790 \times 10^{-7}$ | $8.9700 \times 10^{-8}$ | $8.9700 \times 10^{-8}$ |
| 2 | 0.02 | $7.3990 \times 10^{-7}$ | $4.1440 \times 10^{-7}$ | $4.1450 \times 10^{-7}$ |

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| | | | | |
|---|---|---|---|---|
| 3 | 0.03 | $7.8710 \times 10^{-7}$ | $7.0440 \times 10^{-7}$ | $7.0450 \times 10^{-7}$ |
| 4 | 0.04 | $6.7800 \times 10^{-7}$ | $1.0956 \times 10^{-6}$ | $1.0959 \times 10^{-6}$ |
| 5 | 0.05 | $3.1380 \times 10^{-7}$ | $3.6240 \times 10^{-6}$ | $8.6250 \times 10^{-7}$ |
| 6 | 0.06 | $1.4890 \times 10^{-7}$ | $8.3300 \times 10^{-7}$ | $8.3300 \times 10^{-7}$ |
| 7 | 0.07 | $7.0400 \times 10^{-8}$ | $5.6460 \times 10^{-7}$ | $5.6460 \times 10^{-7}$ |
| 8 | 0.08 | $3.0200 \times 10^{-8}$ | $8.5110 \times 10^{-7}$ | $8.5120 \times 10^{-7}$ |
| 9 | 0.09 | $7.5070 \times 10^{-7}$ | $5.8710 \times 10^{-7}$ | $5.8720 \times 10^{-7}$ |
| 10 | 0.10 | $1.5714 \times 10^{-6}$ | $3.5170 \times 10^{-6}$ | $3.5177 \times 10^{-6}$ |

## 4.0 DISCUSSION

A 4th-step linear multistep method (LMM) was developed as presented in equation (2.13). The method is symmetric, implicit and zero stable but certainly not A-stable. The solution of its first characteristic polynomial in MAPLE reveal that the method has two roots of modulus 1. This shows that the method is weakly stable. On the other hand, equation (2.14) is one of top order methods (TOMs) which were considered as unstable but highly accurate methods. Ordinarily, combining a TOM and a weakly stable LMM cannot produce a method which is capable of approximating stiff ODEs. However, [ 8 and 12] combined the TOMs with other LMMs to form block methods which have been successfully implemented on stiff ODEs

To this end, A pair was formed with equation (2.13), which is a LMM of order 6 and equation (2.14) which is a 6th order TOM and used to construct a one-step implicit block method. Analysis was performed on the block method and it was verified to be A-stable

A-stable, self-stating and capable of generating 10 approximations simultaneously. Three examples on a single, a system of two and a system of three stiff ODEs were used to test the efficiency of the block method. The numerical results showed good approximations of the analytical solutions and also compared favourably with other numerical methods.

## 5.0 CONCLUSION

An implicit block method that is self-starting was developed by combining a 4th-step LMM and a 3-step TOM. The block method is A-stable. Numerical experiments confirmed that the method provides good approximation of stiff ODEs

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